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Quadratic algebra as a 'hidden' symmetry of the Hartmann potential

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Abstract. It is shown that operators, commuting with the Hamiltonian of the Hartmann potential form the quadratic Hahn algebra QH(3). The structure of this algebra and its *finite-dimensional representations are* described. An analysis of these representations is applied to obtain all the relevant physical results: energy spectrum, degree of degeneration and overlap functions.

1. Introduction

An 'accidental' degeneration of the energy levels in the Coulomb problem is indeed a very 'thin' property: almost any additional terms which have been added to this potential destroy the specific Coulomb O(4) symmetry and reduce it to the usual O(3) one.

It is interesting, therefore, that the so-called Hartmann potential [1]

$$U_H = -\alpha/r + \beta/r^2 \sin^2\theta \tag{1.1}$$

does not fully destroy the 'accidental' symmetry. The spherical anisotropy of this potential is not a difficulty, but in fact the integrability of this problem is due to the specific angular dependence of this term.

From this point of view this potential has been studied in [2–4], but the first indication about its symmetry can be found in [5].

A search for the symmetry of this potential has been carried out in [3]. The authors of [3] have shown that SU(2) algebra can be used to describe the energy spectrum. However, SU(2) cannot be used to compute overlap matrix elements between wavefunctions in spherical and parabolic coordinate systems (both these systems allow the separation of variables). This computation has been fulfilled in [4]: overlap matrix elements turn out to be expressible in terms of the hypergeometric function ${}_3F_2(1)$. This function provides an analytic continuation of Clebsh–Gordan coefficients (CGC) from integer values of their parameters to the domain of (arbitrary) real ones. These results indicate the existence of an auxiliary symmetry for the Hartmann potential, similar to the O(4) one for the Coulomb potential.

However, in contrast to the case of the Coulomb potential only three independent integrals of motion exist for the Hartmann potential. This is an important problem as the corresponding finite-dimensional Lie algebra cannot be constructed. The authors of [3] have expressed an opinion that the corresponding symmetry algebra may be an infinite-dimensional one like the Kac–Moody algebra.

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We will show that the symmetry of the Hartmann potential is actually described by quadratic algebra of some special type (we call it the Hahn algebra QH(3), see [6]). This algebra allows one to realize the whole algebraic programme: to find the energy levels and their degeneration degree, overlap matrix coefficients, etc.

2. The symmetry generators

The Schrödinger equation for a particle in the Hartmann potential has the form

$$H\psi = (p^2/2 - \alpha/r + \beta/r^2\sin^2\theta)\psi = E\psi \qquad (\alpha > 0, \beta > 0, E < 0, m = \hbar = 1). \tag{2.1}$$

It is possible to separate the variables in spherical coordinates:

$$\psi(\boldsymbol{r}) = R(r)F(\theta)\exp(im\varphi). \tag{2.2}$$

Despite the appearance of an anisotropic term in the Hartmann potential the degeneration of energy levels is not fully destroyed as might be supposed *a priori*. Thus, an additional symmetry exists and one must find the operators commuting with H and construct from them the corresponding algebra.

The operators commuting with H are known [3]. One of these operators is obvious—it is the modified square of angular momentum:

$$\tilde{L}^2 = L^2 + 2\beta/\sin^2\theta. \tag{2.3}$$

Also, a modified $\tilde{A}_z$ exists:

$$\tilde{A}_z = [\partial_z, \tilde{L}^2/2 - \alpha r] \tag{2.4}$$

where $[\cdot,\cdot]$ denotes the commutator.

The third integral of motion is, of course, L_z (due to azimuthal symmetry) but this operator also commutes with $\tilde{L}^2$ and $\tilde{A}_z$ and may be considered, rather, as a fixed parameter ($L_z = m$).

The other operators like L_x, L_y, A_x, A_y do not commute with the Hamiltonian. So the algebra similar to O(4) does not exist for the Hartmann potential. This property is obvious because of the anisotropy of $U_H(r)$.

However, this is not a problem. We can construct the algebra by means of commutation starting from only two operators:

$$K_0 = \tilde{L}^2 \qquad \text{and} \qquad K_1 = (-2E)^{-1/2}\tilde{A}_z. \tag{2.5}$$

Direct computation yields

$$[K_0, K_1] = K_2 \tag{2.6a}$$

$$[K_0, K_2] = 2(K_0K_1 + K_1K_0) \tag{2.6b}$$

$$[K_2, K_1] = 2K_1^2 + 4K_0 - G \tag{2.6c}$$

where

$$G = 2(m^2 + 2\beta - 1 - \alpha^2/2E) \tag{2.7}$$

is the only parameter of algebra (2.6).

It is seen that commutators of three operators K_0, K_1, K_2 have been closed in the frame of the quadratic algebra. This algebra is a special case of more general structures (see [6, 7]). By quadratic algebra we mean the mathematical objects which have been discovered by Sklyanin in 1983 [8]: the defining property of this algebra is that commutators of generators are expressed in terms of quadratic and linear combinations of generators.

It is worth mentioning that the authors of [3] have supposed that the operators K_0, K_1, $[K_0, K_1]$, $[K_0[K_0, K_1]]$, etc. could form an infinite-dimensional Lie algebra, but have not found an explicit form of this algebra. Our main statement is that the corresponding algebra is not only three-dimensional, but also quadratic.

The only difference between the Hartmann and Coulomb problems is the appearance of the term 2β in the constant G. So the algebra (2.6) is equally applicable to the very well-known case. However, the higher degree of degeneration in the Coulomb potential is really 'accidental" and is due to the existence of auxiliary operators L_x, L_y, A_x, A_y commuting with the Hamiltonian. This will be discussed below.

3. The spectrum, etc.

It is our purpose in this section to derive the main physical results for the Hartmann potential, proceeding only from representations of the QH(3) algebra (2.6).

Let us diagonalize the operator K_0:

$$K_0\psi_p = \lambda_p\psi_p \tag{3.1}$$

(p is a discrete variable with unit step).

Operator K_1 is three diagonal on this basis (for details of calculations see the appendix):

$$K_1\psi_p = a_{p+1}\psi_{p+1} + a_p\psi_{p-1} + b_p\psi_p. \tag{3.2}$$

Combining (3.1) and (3.2) with commutation relations (2.6) one obtains

$$\lambda_p = p(p+1) \tag{3.3a}$$

$$a_p^2 = (p^2 - \xi^2)(\eta^2 - p^2)/(4p^2 - 1) \tag{3.3b}$$

$$b_p = 0 \tag{3.3c}$$

where ξ, η are the roots of the characteristic equation

$$p^4 - (1 + G/2)p^2 + Q/4 = 0. \tag{3.4}$$

Here Q is the value of the Casimir operator. The general expression for the Casimir operator (which commutes with all generators K_α of algebra (2.6)) is

$$Q = -2(K_1^2K_0 + K_0K_1^2) + K_2^2 + 4K_1^2 - 4K_0^2 + 2GK_0 \tag{3.5}$$

and, taking the realization (2.5), it has the value

$$Q = -2\alpha^2(m^2 + 2\beta)/E. \tag{3.6}$$

From (3.4) and (3.6) we obtain

$$\xi^2 = m^2 + 2\beta = M^2 \qquad \eta^2 = -\alpha^2/2E. \tag{3.7}$$

The considered representation must be finite dimensional, being the space of states belonging to the same value of energy. The conditions for this $a_\xi = a_\eta = 0$, together with $a_p^2 > 0$, is fulfilled only if

$$|\xi| \leqslant p \leqslant |\eta| - 1 \tag{3.8}$$

and the dimension of the representation is equal to

$$|\eta|-|\xi|=N=1,2,3,\ldots \tag{3.9}$$

From (3.9) it follows that the energy spectrum is

$$E=-\alpha^2/2(N+M)^2. \tag{3.10}$$

As E does not depend on p, all states with different values of p are degenerate. The degree of degeneration is equal to $2N$ for $m\neq 0$ and N for $m=0$.

If $\beta=0$ (Coulomb case) then additional degeneration arises: E depends only on the principal quantum number $n=N+|m|$ and the degree of degeneration turns out to be

$$s=2\sum_{1}^{n-1} N+n=n^2. \tag{3.11}$$

The reason for this degeneration is the appearance of new integrals L_x, L_y because of spherical symmetry taking place in this case.

In a similar manner one can diagonalize the operator K_1:

$$K_1\varphi_s=\mu_s\varphi_s \tag{3.12}$$

$$K_0\varphi_s=d_{s+1}\varphi_{s+1}+d_s\varphi_{s-1}+h_s\varphi_s. \tag{3.13}$$

Along with similar procedures one obtains

$$\mu_s=2s \qquad h_s=(G-8s^2)/4$$
$$d_s^2=\prod_{i=0}^{3}(s-s_i) \tag{3.14}$$

where the roots s_i are linear combinations of ξ, $\eta>0$:

$$s_0=(1+\xi-\eta)/2 \qquad s_1=(1+\eta-\xi)/2$$
$$s_2=(1+\xi+\eta)/2 \qquad s_3=(1-\xi-\eta)/2. \tag{3.15}$$

The definition region of s is

$$s_0\leqslant s\leqslant s_1-1.$$

The 'length' of this interval s_1-s_0 is the same as for p: $s_1-s_0=N$. It means that the dimension of the $\{\varphi_s\}$ basis coincides with the dimension of the $\{\psi_p\}$ one.

4. Overlap functions and Hahn polynomials

The operator A_z for the Coulomb problem is known to be diagonalized in parabolic coordinates. A similar property is true for the operator $\tilde{A}_z=K_1$ in the case of the Hartmann potential [3]. Our treatment expressed by formulae (3.12)–(3.14) represents a purely algebraic interpretation of this property. By means of this algebra we can (without any concrete realization of operators) obtain overlap functions $\langle\varphi_s|\psi_p\rangle=\langle s|p\rangle$ between two bases—spherical and parabolic.

It is our purpose in this section to derive these overlap functions in terms of the so-called Hahn polynomials [9]. Note that for the Coulomb problem the overlap functions coincide with CGC. The latter is known to be expressed in terms of Hahn polynomials with integer parameters [10].

Let us extract the 'vacuum amplitude' $\psi_0(s)$ from the matrix elements

$$\langle s|p\rangle = \langle s|p_0\rangle P_n(s) \equiv \psi_0(s)P_n(s) \qquad n = p - p_0 = 0, 1, 2, \ldots \tag{4.1}$$

where $p_0 = \xi$ is a minimal value of p and by $P_n(s)$ we denote some function to be established.

Proceeding from the equation

$$\langle s|K_1|p\rangle = 2s\langle s|p\rangle \tag{4.2}$$

one can obtain from (3.2) the recurrent relation for $P(s)$:

$$A_{n+1}P_{n+1}(s) + A_nP_{n-1}(s) = 2sP_n(s) \tag{4.3}$$

where

$$A_n^2 = [(n+\xi)^2 - \xi^2][\eta^2 - (n+\xi)^2]/[4(n+\xi)^2 - 1]. \tag{4.4}$$

The recurrent relation (4.3) (together with conditions $P_0(s) \equiv 1$ and $A_0 = 0$) uniquely determines $P_n(s)$ as the set of polynomials of degree n from argument s. For example,

$$P_1(s) = 2s/A_1 = 2s\sqrt{(2\xi+3)/(\eta+\xi+1)(\eta-\xi-1)}.$$

The polynomials are mutually orthogonal,

$$\delta_{mn} = \langle \psi_m|\psi_n\rangle = \sum_s \langle m|s\rangle\langle s|n\rangle$$
$$= \sum_s w(s)P_m(s)P_n(s) \tag{4.5}$$

where the weight function $w(s)$ is the absolute square of the 'vacuum amplitude'

$$w(s) = |\langle s|p_0\rangle|^2 = |\psi_0(s)^2|. \tag{4.6}$$

One can find the expression for $w(s)$. Indeed, from (3.1) and (3.13),

$$\langle s|K_0|p_0\rangle = p_0(p_0+1)\langle s|p_0\rangle = d_{s+1}\langle s+1|p_0\rangle + d_s\langle s-1|p_0\rangle + h_s\langle s|p_0\rangle \tag{4.7a}$$

$$\langle s|K_2|p_0\rangle = 2(p_0+1)A_1\langle s|p_0+1\rangle = 2d_{s+1}\langle s+1|p_0\rangle - 2d_s\langle s-1|p_0\rangle. \tag{4.7b}$$

Taking into account that

$$A_1\langle s|p_0+1\rangle = 2s\langle s|p_0\rangle$$

follows from (4.3) we have, from (4.7),

$$\frac{\psi_0(s+1)}{\psi_0(s)} = \frac{2p_0(p_0+1) - 2h_s + 4s(p_0+1)}{4d_{s+1}}$$
$$= \sqrt{\frac{(s+1-s_1)(s+1-s_3)}{(s+1-s_0)(s+1-s_2)}}. \tag{4.8}$$

This formula may be considered as the recurrent equation for $w(s)$. Its solution within normalization is

$$w(s) = \Gamma(s+1-s_1)\Gamma(s+1-s_3)/\Gamma(s+1-s_0)\Gamma(s+1-s_2). \tag{4.9}$$

It is well known that the weight function uniquely determines the system of orthogonal polynomials. We can compare $w(s)$ with the weight function of Hahn polynomials $H_n(x; \alpha, \beta, N)$ [9]:

$$w_H(x) = \Gamma(x+\alpha+1)\Gamma(N-x+\beta)/\Gamma(x+1)\Gamma(N-x) \qquad 0 \leqslant x \leqslant N-1. \tag{4.10}$$

They are identical if $s = s_0 + x$, $\alpha = \beta = \xi$, $N = \eta - \xi$.

Thus, we have

$$P_n(s) = H_n(x;\xi,\xi,N) = C_n\,{}_3F_2(-n,-x,n+1+2\xi;\xi+1,1-N|1) \tag{4.11}$$

within the normalization factor, which is not relevant for our purposes.

Now for the overlap function we obtain the following result:

$$\langle\varphi_s|\psi_p\rangle = \psi_0(s)H_{p-p_0}(s-s_0). \tag{4.12}$$

The expression for $\langle\varphi_s|\psi_p\rangle$ in terms of ${}_3F_2(1)$ has been obtained in [4] by direct computation. However, the authors of [4] have not noticed the mere coincidence of their result with the Hahn polynomial. Our purely algebraic approach gives the reason for the appearance of this: it is the representation of 'hidden' quadratic symmetry of the Hartmann potential.

In conclusion let us make some remarks.

The algebra with commutation relations (2.6) is a special case of the complete Hahn algebra QH(3) (see [6, 7]):

$$\begin{aligned}
&[K_0, K_1] = K_2\\
&[K_0, K_2] = 2\{K_0, K_1\} + C_1K_1 + DK_0 + G_1\\
&[K_2, K_1] = 2K_1^2 + C_0K_0 + DK_1 + G_0
\end{aligned} \tag{4.13}$$

with arbitrary real parameters C_0, C_1, D, G_0, G_1 (here the considered case corresponds to $C_1 = D = G_1 = 0$).

For our value of parameter $C_0 = 4$ the spectrum of K_1 is discrete and as a result we obtained the Hahn polynomials of discrete argument—CGC, extended to real values of their arguments. One can say that algebra (4.13) forms a foundation for the usual Clebsch–Gordan scheme in the theory of angular momentum.

For $C_0 < 0$ the spectrum of K_1 is continuous and we would be dealing with Hahn polynomials of continuous argument. These polynomials are so-called Lorentz polynomials considered earlier in [11].

Finally, if $C_0 = 0$ we obtain the quadratic Jacobi algebra QJ(3), which is the dynamical symmetry algebra for exactly solvable one-dimensional potentials [12].

5. Conclusion

It was the main purpose of this paper to construct the algebra corresponding to the symmetry of the Hartmann potential. We hope that the reader has been convinced that the somewhat unusual object—quadratic algebra QH(3)—allows one to present a completely algebraic solution of the problem. It must be noted that the authors of [3] have proposed an idea closely related to our approach. However, due to their paradigm (namely, the search for an algebra with linear commutators like Virasoro or Kac–Moody algebras) they have not noticed that the right-hand sides of the commutation relations are quadratic combinations of the initial operators, i.e. are closed in the frame of quadratic algebra.

We hope this simple solution of the symmetry problem will draw the attention of physicists to the quadratic algebras which have been successfully used in statistical mechanics [8], the theory of 6-j symbols [7], exactly solvable potentials in quantum mechanics [12] and in other areas.

There is one interesting and as yet unresolved problem: what group corresponds to the quadratic dynamical algebra? In the Coulomb problem the O(4) Lie group is the corresponding group acting on the four-dimensional sphere in momentum space [13]. However, we have not found any similar group in the case of the Hartmann potential.

Appendix 1. How to find the spectrum of generators

Let ψ_p be the eigenvalue basis for K_0:

$$K_0\psi_p=\lambda_p\psi_p \qquad K_1\psi_p=\sum_q A_{pq}\psi_q. \tag{A1}$$

Substituting (A1) into the (2.6*b*) one obtains

$$A_{pq}[(\lambda_p-\lambda_q)^2-2(\lambda_p+\lambda_q)]=0. \tag{A2}$$

If $q=p$ we have

$$A_{pp}=0. \tag{A3}$$

If $q\neq p$ there exists only two different values of q (with p fixed) as the roots of quadratic equation (A2).

So the matrix A_{pq} is three diagonal and we can choose

$$A_{pq}=a_{p+1}\delta_{p,q-1}+a_p\delta_{p,q+1} \tag{A4}$$

in accordance with (3.2) ($b_p=A_{pp}=0$).

The expression for the spectrum λ_p is obtained from (A2) if $q=p\pm1$:

$$\lambda_p=p(p+1) \tag{A5}$$

with an undetermined initial value p_0 of p,

$$p=p_0+n \qquad n=0,1,2,\ldots.$$

Notice that the quadratic spectrum (A5) is due to bilinearity of commutation relation (2.6*b*).

Appendix 2. How to calculate the matrix element a_p

Matrix element a_p can be found from the commutation relation (2.6*c*)

$$\langle p|[K_2,K_1]|p\rangle=2\langle p|K_1^2|p\rangle+4\lambda_p-G. \tag{A6}$$

From (3.2) we have $\langle p|K_1^2|p\rangle=a_p^2+a_{p+1}^2$. The action of operator K_2 on the basis ψ_p is

$$K_2\psi_p=2(p+1)a_{p+1}\psi_{p+1}-2pa_p\psi_{p-1}. \tag{A7}$$

Thus

$$\langle p|[K_2,K_1]|p\rangle=4(pa_p^2-(p+1)a_{p+1}^2)$$

and

$$2(2p+3)a_{p+1}^2-2(2p-1)a_p^2=G-4p(p+1). \tag{A8}$$

On the other hand, from the expression for the Casimir operator (3.5) we have

$$Q=\langle p|K_2^2-2(K_0K_1^2+K_1^2K_0)+2GK_0+4K_1^2-4K_0^2|p\rangle$$
$$=2G\lambda_p-4\lambda_p^2-4[p(2p+3)a_{p+1}^2+(p+1)(2p-1)a_p^2]. \tag{A9}$$

Combining (A8) and (A9), one obtains

$$a_p^2=\frac{-p^2(p^2-1)+Gp^2/2-Q/4}{4p^2-1}=\frac{(p^2-\xi^2)(\eta^2-p^2)}{4p^2-1}. \tag{A10}$$

In a similar way one can obtain the formulae (3.13) and (3.14) for the basis φ_ς.

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